MATH2060 TUTO 8

8. Let $F(x)$ be defined for $x \geq 0$ by $F(x) := (n-1)x - (n-1)n/2$ for $x \in [n-1, n)$, $n \in \mathbb{N}$. Show that F is continuous and evaluate $F'(x)$ at points where this derivative exists. Use this result to evaluate $\int_a^b [\![x]\!]dx$ for $0 \leq a < b$, where $[\![x]\!]$ denotes the greatest integer in x, as defined in Exercise 5.1.4.

Ans: To show that f is continuous, it suffices to check that

$$\lim_{x \to n^-} F(x) = F(n) \qquad \forall n \in \mathbb{N}.$$

(clearly $\lim_{x \to n^+} F(x) = F(n)$)

Indeed, $\lim_{x \to n^-} F(x) = \lim_{x \to n^-} \left[(n-1)x - (n-1)n/2 \right] = (n-1)n - (n-1)n/2 = \frac{(n-1)n}{2}$

$$F(n) = (n)n - n(n+1)/2 = \frac{n}{2}(2n-n-1) = \frac{(n-1)n}{2}$$

So f is cts on $[0, \infty)$.

$\forall n \in \mathbb{N}$,

$$\lim_{x \to n^-} \frac{F(x) - F(n)}{x - n} = \lim_{x \to n^-} \frac{(n-1)x - (n-1)n/2 - (n-1)n/2}{x - n}$$

$$= \lim_{x \to n^-} \frac{(n-1)(x-n)}{x-n} = n-1.$$

$$\lim_{x \to n^+} \frac{F(x) - F(n)}{x - n} = \lim_{x \to n^+} \frac{nx - n(n+1)/2 - (n-1)n/2}{x-n}$$

$$= \lim_{x \to n^+} \frac{n(x-n)}{x-n} = n$$

So F is not diff. at any $n \in \mathbb{N}$.

Also $F'(x) = n-1 = [\![x]\!]$ for $x \in (n-1, n)$, $n \in \mathbb{N}$.

$F'(0) = 0 = [\![0]\!]$ (right derivative)

Now a) F is cts on $[a,b]$

b) $F'(x) = [\![x]\!] \quad \forall x \in [a,b] \setminus E$,

where $E := [a,b] \cap \mathbb{N}$ is a finite set

c) $[\![x]\!] \in \mathcal{R}[a,b]$ since it is a step fcn.

By FTC (1st form),

$$\int_a^b [\![x]\!]\,dx = F(b) - F(a)$$

$$= \left([\![b]\!]b - [\![b]\!]([\![b]\!]+1)/2 \right) - \left([\![a]\!]a - [\![a]\!]([\![a]\!]+1)/2 \right)$$

14. Show there does not exist a continuously differentiable function f on $[0,\ 2]$ such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \le 2$ for $0 \le x \le 2$. (Apply the Fundamental Theorem.)

Ans: Suppose f is such a fcn.

Then a) f is cts on $[0,2]$ (since f is diff. on $[0,2]$)

b) $f'(x) = f'(x)$ $\forall x \in [0,2]$ (trivial)

c) $f' \in \mathcal{R}[0,2]$ (since f' is cts on $[0,2]$)

By FTC (1st form),

$$f(2) - f(0) = \int_0^2 f'(x)\,dx$$

$$\Rightarrow \quad 4 - (-1) \le \int_0^2 2\,dx$$

$$\Rightarrow \quad 5 \le 4$$, impossible.

So there is no such f. //

7.3.8 Substitution Theorem *Let $J := [\alpha, \beta]$ and let $\varphi : J \to \mathbb{R}$ have a continuous derivative on J. If $f : I \to \mathbb{R}$ is continuous on an interval I containing $\varphi(J)$, then*

$$\int_{\alpha}^{\beta} f(\varphi(t)) \cdot \varphi'(t)\,dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx. \tag{5}$$

17. Use the following argument to prove the Substitution Theorem 7.3.8. Define $F(u) := \int_{\varphi(\alpha)}^{u} f(x)dx$ for $u \in I$, and $H(t) := F(\varphi(t))$ for $t \in J$. Show that $H'(t) = f(\varphi(t))\varphi'(t)$ for $t \in J$ and that

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)dx = F(\varphi(\beta)) = H(\beta) = \int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t)dt.$$

Ans: Since f is cts on $I := [a,b]$, FTC (2nd form) implies that

$$F(u) = \int_{\varphi(\alpha)}^{u} f(x)\,dx \quad \text{is diff. on } [a,b]$$

and $F'(u) = f(u) \quad \forall\, u \in [a,b]$ (Note $\varphi(\alpha) \in I$)

By Chain Rule, $H = F \circ \varphi$ is diff. on J

and $H'(t) = F'(\varphi(t)) \cdot \varphi'(t) \quad \forall\, t \in J$

Now, a) H is cts on $[\alpha, \beta]$

b) $H'(t) = F'(\varphi(t)) \cdot \varphi'(t)$

$= f(\varphi(t)) \cdot \varphi'(t) \quad \forall\, t \in [\alpha, \beta]$

c) $(f \circ \varphi) \cdot \varphi' \in \mathcal{R}[\alpha, \beta]$

since it is cts on $[\alpha, \beta]$

By FTC (1st form),

$$\int_{\alpha}^{\beta} f(\varphi(t)) \cdot \varphi'(t)\,dt = H(\beta) - H(\alpha) = F(\varphi(\beta)) - \cancelto{0}{F(\varphi(\alpha))}$$

i.e. $$\int_{\alpha}^{\beta} f(\varphi(t)) \cdot \varphi'(t)\,dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx$$

§7.4

3. Let f and g be bounded functions on $I := [a,\, b]$. If $f(x) \le g(x)$ for all $x \in I$, show that $L(f) \le L(g)$ and $U(f) \le U(g)$.

Ans: Suppose $|f(x)|, |g(x)| \le M \quad \forall\, x \in [a,b]$.

Let $P = (x_0, x_1, \dots, x_n)$ be a partition of $[a,b]$.

Then, $\forall\, k = 1, \dots, n$,

$$-M \le f(x) \le g(x) \le M \quad \forall\, x \in [x_{k-1}, x_k].$$

$$\Rightarrow -M \le \inf_{[x_{k-1}, x_k]} f \le \inf_{[x_{k-1}, x_k]} g \le M$$

So $$\sum_{k=1}^{n} \inf_{[x_{k-1}, x_k]} f\,(x_k - x_{k-1}) \le \sum_{k=1}^{n} \inf_{[x_{k-1}, x_k]} g\,(x_k - x_{k-1}) \le M(b-a)$$

i.e. $$L(f;P) \le L(g;P) \le M(b-a) \qquad (*)$$

Since $(*)$ is true for any $P \in \mathcal{P}([a,b])$. ← set of all partitions of $[a,b]$.

we have $L(f) := \sup\{L(f;P) : P \in \mathcal{P}([a,b])\}$

$L(g) := \sup\{L(g;P) : P \in \mathcal{P}([a,b])\}$ both exist

and

$$L(f) \le L(g)$$

Similarly, we can show that

$$U(f) \le U(g)$$

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5. Let f, g, h be bounded functions on $I := [a,b]$ such that $f(x) \le g(x) \le h(x)$ for all $x \in I$. Show that if f and h are Darboux integrable and if $\int_a^b f = \int_a^b h$, then g is also Darboux integrable with $\int_a^b g = \int_a^b f$.

Ans: Since f and h are Darboux integrable, we have

$$L(f) = U(f) = \int_a^b f$$
$$L(h) = U(h) = \int_a^b h$$

By Ex 3, $f(x) \le g(x) \le h(x)$ $\forall x \in [a,b]$ implies that

$$L(f) \le L(g) \le L(h)$$
$$U(f) \le U(g) \le U(h)$$

Since $\int_a^b f = \int_a^b h$, we have

$$L(g) = U(g) = \int_a^b f = \int_a^b h.$$

Hence g is Darboux integrable with

$$\int_a^b g = \int_a^b f = \int_a^b h$$

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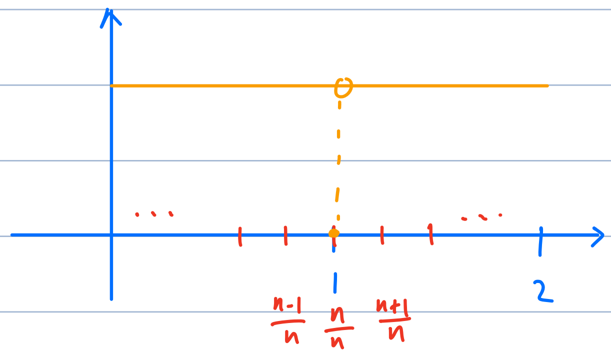
6. Let f be defined on $[0, 2]$ by $f(x) := 1$ if $x \neq 1$ and $f(1) := 0$. Show that the Darboux integral exists and find its value.

Ans: $\forall n \in \mathbb{N}$, let P_n be the uniform partition of $[0,2]$ into $2n$ subintervals given by $P_n := \left(0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{2n-1}{n}, \underbrace{\frac{2n}{n}}_{=2}\right)$.



Then,

$$m_k = \inf_{[\frac{k-1}{n}, \frac{k}{n}]} f = 1$$

$$M_k = \sup_{[\frac{k-1}{n}, \frac{k}{n}]} f = 1 \qquad \forall k \in \{1, \dots, 2n\} \setminus \{n, n+1\},$$

and

$$\inf_{[\frac{n-1}{n}, \frac{n}{n}]} f = \inf_{[\frac{n}{n}, \frac{n+1}{n}]} f = 0, \quad m_n = m_{n+1} = 0$$

$$\sup_{[\frac{n-1}{n}, \frac{n}{n}]} f = \sup_{[\frac{n}{n}, \frac{n+1}{n}]} f = 1, \quad M_n = M_{n+1} = 1$$

Moreover, $x_k - x_{k-1} = \frac{1}{n}$.

So,

$$L(f; P_n) = (2n-2)(1)\frac{1}{n} + 2(0)\frac{1}{n} = 2 - \frac{2}{n}$$

$$U(f; P_n) = (2n-2)(1)\frac{1}{n} + 2(1)\frac{1}{n} = 2$$

Hence, we find a seq $\{P_n\}$ of partition of $[0,2]$ s.t.

$$\lim_{n\to\infty} \left(U(f; P_n) - L(f; P_n)\right) = \lim_{n\to\infty} \frac{2}{n} = 0$$

By Corollary 7.49, f is Darboux integrable and

$$\int_0^2 f\,dx = \lim_{n\to\infty} L(f; P_n) = \lim_{n\to\infty} U(f; P_n) = 2$$

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15. Let f be defined on $I := [a, b]$ and assume that f satisfies the Lipschitz condition $|f(x) - f(y)| \leq K|x - y|$ for all x, y in I. If $\mathcal{P}_n$ is the partition of I into n equal parts, show that $0 \leq U(f; \mathcal{P}_n) - \int_a^b f \leq K(b-a)^2/n$.

Ans: Write $P_n = (x_0, x_1, \ldots, x_n)$

Since f is Lipschitz, hence cts on $[a, b]$, $\int_a^b f$ exists

Moreover,

$\exists\ t_i \in [x_{i-1}, x_i]$ s.t. $f(t_i) = M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x)$

$\exists\ s_i \in [x_{i-1}, x_i]$ s.t. $f(s_i) = m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x)$

By def., $L(f; P_n) \leq \int_a^b f \leq U(f; P_n)$.

Thus,

$$\begin{aligned}
0 \leq U(f; P_n) - \int_a^b f &\leq U(f; P_n) - L(f; P_n) \\
&= \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) \\
&= \sum_{i=1}^n (f(t_i) - f(s_i))(x_i - x_{i-1}) \\
&\leq \sum_{i=1}^n K|t_i - s_i|(x_i - x_{i-1}) \\
&\leq K \sum_{i=1}^n (x_i - x_{i-1})^2 \\
&= K \sum_{i=1}^n \left(\frac{b-a}{n}\right)^2 \\
&= K(b-a)^2/n
\end{aligned}$$

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